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Area Distribution for Directed Random Walks

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We study the probability distribution for the area under a directed random walk in the plane. The walk can serve as a simple model for avalanches based on the idea that the front of an avalanche can be described by a random walk and the size is given by the area enclosed. This model captures some of the qualitative features of earthquakes, avalanches, and other self-organized critical phenomena in one dimension. By finding nonlinear functional relations for the generating functions we calculate directly the exponent in the size distribution law and find it to be 4/3.

KEY WORDS: Directed random walks.

1. INTRODUCTION

The properties of directed random walks (DRW) are of interest for their own sake and also because many other physical problems can be mapped onto them, at least in some approximation. In this paper we compute various properties of an ensemble of directed random walks which arises, for example, from a simple mean-field-like approach to avalanches.

The ensemble consists of walks in $\mathbb{R}^2$ between points with integer coordinates which start at the origin and are confined to the first quadrant. At every step, i, the x coordinate increases by one, $x_{i+1} = x_i + 1$. The y coordinate may increase by one, $y_{i+1} = y_i + 1$, with weight α; stay the same, $y_{i+1} = y_i$, with weight β; or decrease by one, $y_{i+1} = y_i - 1$, with weight α. If the value of the y coordinate at the ith step is zero then the walk stops (the very first step must therefore be from $(0, 0)$ to $(1, 1)$). For most of this paper we will work with $\alpha = 1$, $\beta = 0$; however the more

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general case is solvable by precisely the same methods and enables us to study the universal nature of the exponents. In addition we will consider the case where there is a reflecting barrier at $y = h$; if the walk hits the barrier at the ith step then the next step must be to decrease y by one.

Within this ensemble it is natural to enquire about the number of walks satisfying various conditions (or, equivalently, the probability that a given walk satisfies those conditions). We define the duration of the walk, T, to be that (non-zero) value of i for which $y_i = 0$ and the area, A, by

$$A = \sum_{i=1}^{T-1} y_i \tag{1}$$

The probability $P(T)$ that a walk has duration T is the classic gambler's ruin problem which was first solved by Lagrange.[1] We are mainly concerned with the probability $P(A)$ that a walk has area A. This problem would doubtless also have been solved by Lagrange if the authorities had levied a tax on a player's integrated holding over a game. We will show explicitly that in the absence of a reflecting barrier

$$P(A) \sim A^{-4/3} \tag{2}$$

for large enough A, a result that is expected on scaling grounds. In the presence of a reflecting barrier the dependence becomes exponential for large enough A.

Our method is to derive a nonlinear functional equation for the generating function for the number of walks with a fixed duration covering a given area. After completing this work we discovered that our method is quite similar to the one recently introduced for the study of various polygon problems by Prellberg and Brak.[2] Their polygon problems are expected to be in the same universality class as the one studied here. The nonlinear functional equation for the generating function has a solution in terms of q-series. However, it is not easy to extract the critical exponents from the q-series so we adopt a more direct approach. A similar problem was studied by Abraham and Smith[3] in relation to a simple model of wetting, and in continous time by Louchard[4] where the generating function for the Brownian excursion area density is determined.

The ensemble of walks we study is related to a simple model for the propagation of a one-dimensional avalanche front which was suggested by studying earthquakes in the Burridge–Knopoff model.[5–9] The basic idea is to assume that neighbouring parts of an avalanche front are correlated in the following way. If we label elements of the avalanche by integers i and an element i moves a distance h_i then the displacement of its neighbour, labelled by $i+1$, is distributed with a probability distribution $P_i(h_{i+1}) = \phi(h_{i+1} - h_i)$ which is centered on h_i but otherwise independent of i. This distribution is modified by an appropriate boundary condition at $h_{i+1} = 0$ amounting to a certain killing probability for the avalanche. The simplest possible probability distribution ϕ corresponds to the avalanche front performing a Bernoulli random walk on the positive integers and terminating once it returns to 0; this is precisely the DRW ensemble defined above with $\alpha = 1$, $\beta = 0$. The scaling law relating the frequency of avalanches, $N_w(k)$, to their width (= number of sites that move) k, is

$$N_w(k) \sim k^{-3/2} \tag{3}$$

and the frequency, $N_s(A)$, of avalanches of size A is

$$N_s(A) \sim A^{-4/3} \tag{4}$$

The introduction of a mean field theory for self-organized criticality[10] is quite natural and has been done in many different ways for different models yielding identical critical exponents.[11–14] The exponent 4/3 has been derived by simple scaling arguments in a model of avalanches related to the one we have just described.[15] Sometimes, for example in earthquake models, it is natural to place an upper bound on the maximal allowed displacement h_i and in that case the power law turns into an exponential decay for large A. The relation to earthquakes, in which (4) is the Gutenberg–Richter law,[16] is discussed in more detail in ref. 17.

2. SHORT-TIME BEHAVIOUR

We begin by considering the DRW ensemble for $\alpha = 1$, $\beta = 0$, in the absence of the reflecting barrier. Let $\mathscr{W}$ denote the set of all such walks and let $N(A, T)$ denote the number of walks in $\mathscr{W}$ of duration T and area A (1). Let

$$N(T) = \sum_A N(A, T) \tag{5}$$

The probability that the walk returns to the origin after T steps is given by[1]

$$\begin{aligned} P(T) &= 2^{-T+1} N(T) \\ &= \frac{2^{-T+1}}{T-1} \binom{T-1}{T/2} \end{aligned} \tag{6}$$

if T is even and 0 otherwise (we divide by 2^{T-1} since the first step in the walk is given). For large T

$$P(T) \sim T^{-3/2} \tag{7}$$

Similarly, the conditional probability $P(A \mid T)$ that a walk covers an area A, given that it lasts a time T, can be written as

$$P(A \mid T) = \frac{N(A, T)}{N(T)} \tag{8}$$

It follows that the probability of area A is given by

$$P(A) = 2^{-T+1} \sum_T N(A, T) \tag{9}$$

It is natural to expect the average height $\langle h_n \rangle$ of a walk which lasts a time T to scale like $\sqrt{T}$ for large T and hence the average area $\langle A \rangle \sim T^{3/2}$ for such walks. If we drop the constraint that the walk ends when it hits the x axis and define the area under the walks to be positive if the walk is in the upper half plane and negative when it is in the lower half plane, we can express the area as a sum of independent but non-identical random variables. The generalized central limit theorem[1] applies to this sum and we find that asymptotically the area is normally distributed around zero with a variance T^3. Assuming that $P(A \mid T)$ is normally distributed around $\langle A \rangle$ with the same variance, T^3, we expect that for large A

$$\begin{aligned} P(A) &= \sum_T P(A \mid T)\, P(T) \\ &\sim \int_0^\infty T^{-3} \exp\left(\frac{(A - T^{3/2})^2}{T^3}\right) dT \\ &\sim A^{-4/3} \end{aligned} \tag{10}$$

Below we shall verify that the asymptotic behaviour of $P(A)$ is indeed given by (10) even though one can prove that $P(A \mid T)$ is not normally distributed by computing its first few moments.

To prove (10) we begin by finding an equation for the generating function of walks. Let $\tilde{\mathscr{W}}$ denote the class of directed walks in $\mathscr{W}$ which avoid the line $y = 1$ until they return to $y = 0$, i.e., if $w \in \tilde{\mathscr{W}}$ and w is of duration T then $y_i > 1$ for $1 < i < T-2$. Now denote by $\tilde{N}(A, T)$ the number of paths in $\tilde{\mathscr{W}}$ which return to 0 at time T and cover an area A. Then

$$\tilde{N}(A, T) = N(A - T + 1, T - 2) \tag{11}$$

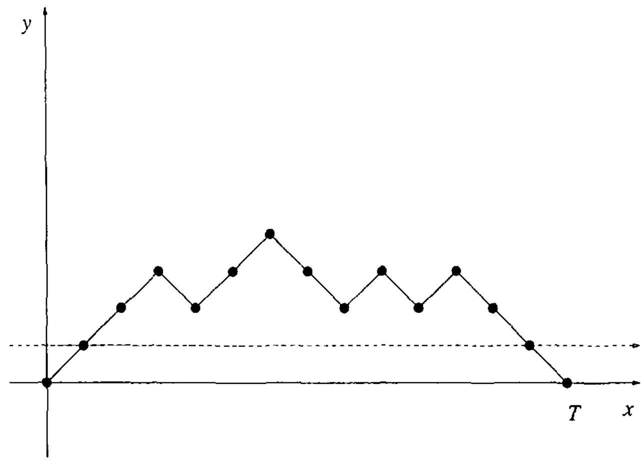

Fig. 1. This figure illustrates the one to one correspondence between paths in $\tilde{\mathscr{W}}$ of duration T and paths in $\mathscr{W}$ of duration $T-2$.

(see Fig. 1). Now consider any directed walk $w \in \mathscr{W}$ which lasts a time $T > 2$ and covers an area A. Let T_1 denote the smallest integer > 1 such that $y_{T_1} = 1$. The largest possible value of T_1 is of course $T_1 = T - 1$. If we cut the path w in two pieces at the point $(T_1, 1)$ then we can associate uniquely to w two paths, $\tilde{w} \in \tilde{\mathscr{W}}$ and $w_1 \in \mathscr{W}$, of duration $T_1 + 1$ and $T - T_1 + 1$, respectively, see Fig. 2. In the extreme case $T_1 = T - 1$ the

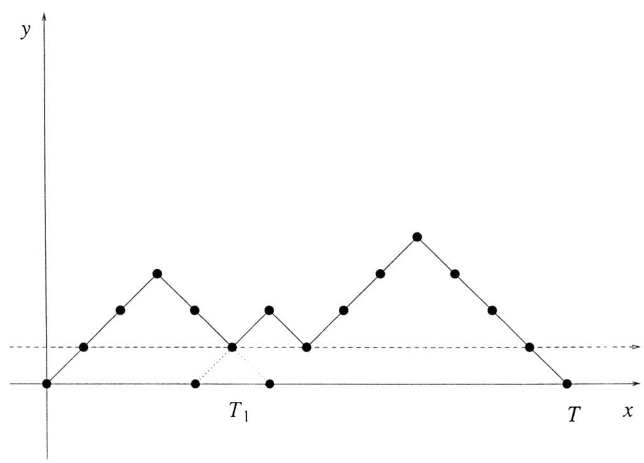

Fig. 2. This figure illustrates the unique decomposition of any directed path into a pair of paths in $\tilde{\mathscr{W}}$ and $\mathscr{W}$.

second walk is the trivial one of length 2. If we denote the area under the first walk by A_1 then the area under the second one equals $A - A_1 + 1$ and we find that

$$N(A, T) = \delta_{A1}\,\delta_{T2} + \sum_{T_1=1}^{T-1} \sum_{A_1=1}^{A} \tilde{N}(A_1, T_1+1)\,N(A - A_1 + 1, T - T_1 + 1)$$
$$= \delta_{A1}\,\delta_{T2} + \sum_{T_1=1}^{T-1} \sum_{A_1=1}^{A} N(A_1 - T_1, T_1 - 1)$$
$$\times N(A - A_1 + 1, T - T_1 + 1) \qquad (12)$$

by (11). The first term on the right side of (12) corresponds to the two step path. We define the generating function $f(z, u)$ for the numbers $N(A, T)$ by

$$f(z, u) = \sum_{T=2}^{\infty} \sum_{A=1}^{\infty} N(A, T)\, z^A u^T \qquad (13)$$

which is convergent for $|z| < 1$ and $|u| \leqslant \frac{1}{2}$. The recurrence relation (12) can now be rewritten as the non-linear functional equation

$$f(z, u) = zu^2 + f(z, u)\, f(z, uz) \qquad (14)$$

It is straightforward to extend this derivation to general weights α, β. Letting

$$g(z, u) = \sum_{w} \alpha^{2n_+(w) - 1} \beta^{T - 2n_+(w)} z^A u^T \qquad (15)$$

where $n_+(w)$ is the number of steps in w for which y increases, and the sum runs over walks which may remain at a constant y-coordinate, we find

$$g(z, u) = \frac{\alpha}{1 - \beta uz}\,(zu^2 + g(z, u)\, g(z, uz)) \qquad (16)$$

The equations (14) and (16) can be linearized by the method of ref. 2. For example, if we use the substitution

$$f(z, u) = zu^2\,\frac{H(z, zu)}{H(z, u)} \qquad (17)$$

in (14) we obtain

$$z^3u^2 H(z, z^2u) - H(z, zu) + H(z, u) = 0 \qquad (18)$$

which has the q-series solution

$$H(z, u) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n(2n+1)} u^{2n}}{(z; z)_n (-z; z)_n} \tag{19}$$

where

$$(y; z)_n = \prod_{m=0}^{n-1} (1 - yz^m) \tag{20}$$

The generating functions for polygon problems were first obtained in terms of q-series[18, 19] by solving the coupled *linear* eqations for the generating functions found using Temperley's method.[20] The analysis of the q-series solution to extract critical exponents is complicated; for the polygon problems this was tackled in ref. 2 and, more completely, by Prellberg[21] who found an explicit form for the scaling function which can be compared directly with the continuous time calculations in ref. 4. However, as we will show, it is straightforward to obtain critical exponents by working directly with the nonlinear functional equation for the generating function and avoiding the q-series analysis entirely.

For $z = 1$ we can easily solve (14) and find

$$f(1, u) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4u^2} \tag{21}$$

which for $u = \frac{1}{2}$ takes the value $\frac{1}{2}$ in accordance with (9); the square root singularity implies the asymptotic behaviour (7) for $P(T)$. For general values of z and u (14) and (16) have explicit solutions in terms of q-series but they can also be rearranged to yield a continued fraction; for example (14) yields upon iteration the continued fraction for $f(z, u)$

$$f(z, u) = \cfrac{zu^2}{1 - \cfrac{z^3u^2}{1 - \cfrac{z^5u^2}{1 - \cdots}}} \tag{22}$$

This function has a natural boundary at $|z| = 1$ meaning that it cannot be analytically continued beyond the unit circle; such behaviour is generic for area generating functions in polygon problems.[2]

Looking at Eq. (10) we expect the average area $\langle A \rangle$ to diverge, i.e., we expect

$$\lim_{u \uparrow 1/2} \frac{\partial}{\partial z} f(z, u) \bigg|_{z=1} = \infty \tag{23}$$

Indeed, we shall prove that

$$\frac{\partial^n}{\partial z^n} f(z, u)\bigg|_{z=1} \sim (1-4u^2)^{(1/2)-(3n/2)} \tag{24}$$

as $u \uparrow \frac{1}{2}$, for any $n \geqslant 1$. Let us define

$$P_u(A) = \sum_T u^{T-1} N(A, T) \tag{25}$$

Then

$$\sum_A A^n P_u(A) = \left(z \frac{\partial}{\partial z}\right)^n f(z, u)\bigg|_{z=1} \tag{26}$$

For any path of duration T, covering an area A, it is easy to see that

$$\tfrac{3}{2}T - 2 \leqslant A \leqslant \tfrac{1}{4}T^2 \tag{27}$$

The lower bound is obtained by considering the path which zigzags between 1 and 2 and covers the smallest possible area while the upper bound corresponds to the triangular path that climbs to height $T/2$ in time $T/2$ and then descends to zero in time $T/2$. It follows that for u smaller than but close to $\frac{1}{2}$ we have the bounds

$$e^{-c_1(1-2u)A} P(A) \leqslant P_u(A) \leqslant e^{-c_2(1-2u)\sqrt{A}} P(A) \tag{28}$$

where c_1 and c_2 are positive constants. It is natural to regard u as a temperature-like parameter and $u = \frac{1}{2}$ as a critical point so we expect scaling as this point is approached. Assuming that

$$P_u(A) \sim A^{-\gamma} F(\sqrt{1-4u^2}\, A^{\beta}) \tag{29}$$

for large A, where F is a function decaying more rapidly than any power, we find that

$$\sum_A A^n P_u(A) \sim (1-4u^2)^{(\gamma-n-1)/2\beta} \tag{30}$$

Since this is valid for any n it follows from Eq. (24) that

$$\beta = \tfrac{1}{3} \qquad \text{and} \qquad \gamma = \tfrac{4}{3} \tag{31}$$

This is of course consistent with the inequalities (28) which imply that $\frac{1}{4} \leqslant \beta \leqslant \frac{1}{2}$.

In order to verify Eq. (24) let us denote the derivatives of the generating function f with respect to the first and second argument by $\partial_1 f$ and $\partial_2 f$, respectively. We begin by considering the case $n=1$. Differentiating Eq. (14) with respect to z we obtain

$$\partial_1 f(z,u) = u^2 + f(z,uz)\,\partial_1 f(z,u) + f(z,u)(\partial_1 f(z,uz) + u\,\partial_2 f(z,uz)) \tag{32}$$

Putting $z=1$, rearranging and using Eq. (21) we obtain

$$\begin{aligned}\partial_1 f(1,u) &= \frac{u^2 + u\,\partial_2 f(1,u)}{1-2f(1,u)}\\ &= \frac{u^2 + 2u^2(1-4u^2)^{-1/2}}{\sqrt{1-4u^2}}\\ &\sim \frac{1}{1-4u^2}\end{aligned} \tag{33}$$

Assume now that Eq. (24) holds for $n \leqslant N-1$ where $N \geqslant 2$. Differentiating Eq. (14) N times with respect to z yields

$$\partial_1^N f(1,u) = \sum_{k=0}^{N} \binom{N}{k} \partial_1^k f(1,u) \sum_{j=0}^{N-k} \binom{N-k}{j} u^{N-j-k}\,\partial_1^j\,\partial_2^{N-j-k} f(1,u) \tag{34}$$

Rearranging we find that

$$\begin{aligned}\partial_1^N f(1,u) = \frac{1}{1-2f(1,u)} &\sum_{k=0}^{N-1} \binom{N}{k} \partial_1^k f(1,u)\\ &\times \sum_{j=0,\, j\neq N}^{N-k} \binom{N-k}{j} u^{N-j-k}\,\partial_1^j\,\partial_2^{N-j-k} f(1,u)\end{aligned} \tag{35}$$

By the inductive hypothesis the most singular terms on the right hand side of Eq. (35) correspond to $j+k=N$ if $k>0$ and $j=N-1$ when $k=0$. The desired result follows.

Working slightly harder we can determine the coefficient of the leading divergence of the Nth moment of $P_u(A)$ and this allows us to place a further restriction on the function F introduced above. In view of Eq. (24) one can write

$$\partial_1^N f(1,u) = C_N(1-4u^2)^{(1/2)-(3N/2)} + O((1-4u^2)^{1-(3N/2)}) \tag{36}$$

It is straightforward to check that Eq. (35) determines the following recursion relation for the coefficients C_N

$$C_N = \sum_{k=1}^{N-1} \binom{N}{k} C_k C_{N-k} + (3N-4)\, N C_{N-1} \tag{37}$$

It follows that up to power corrections

$$C_N \sim (2N)! \tag{38}$$

for large N.

Suppose now that the function F in (29) is an exponential function of a power, i.e.,

$$F(x) = e^{-\alpha x^q} \tag{39}$$

for some constants $\alpha, q > 0$. Using the ansatz (29) to calculate the Nth moment of the area distribution we find that q is fixed to equal 3/2 by the asymptotic formula (38). We therefore expect that

$$P_u(A) \sim A^{-4/3} e^{-\alpha(1-4u^2)^{3/4} A^{1/2}} \tag{40}$$

in agreement with the bound (28).

Similar, although slightly more messy, manipulations establish the same results for the generalized generating function $g(z, u)$ (16). This completes our disussion of the distribution of DRWs where we do not need to take the upper cutoff h into account. So, for example, the exponent 4/3 governs the size distribution of *small* avalanches in the presence of a cutoff as claimed in Section 1.

3. LONG-TIME BEHAVIOUR

We now consider a directed random walk with a reflecting barrier at height $y = h$. Let $p_i(t)$ be the probability of the walk being at height i after t steps; then the initial condition is $p_i(1) = \delta_{i1}$. We let $p(t)$ denote the column vector whose ith entry is $p_i(t)$. Then

$$p(t) = (\tfrac{1}{2} M)^{t-1}\, p(1) \tag{41}$$

where M is a tridiagonal matrix with $M_{i,i} = 0$, $M_{2,1} = 2$, $M_{h,h+1} = 0$ and all other elements on the upper and lower diagonals equal to one. The probability that a walk lasts exactly T steps is evidently

$$P(T) = \tfrac{1}{2} p_1(T-1) \tag{42}$$

Letting e_i, $i=0,\ldots,h$, denote the standard orthonormal basis on $\mathbb{R}^{h+1}$, we find

$$P(T)=\langle e_0, (\tfrac{1}{2}M)^{T-1} e_1\rangle \tag{43}$$

Let D denote the matrix whose elements are defined by

$$D_{ij}=z^i\,\delta_{ij} \tag{44}$$

$i, j=0,\ldots, h$. The probability $P(A, T)$ that a walk lasts a time T and covers an area A is given by the coefficient of z^A in the matrix element

$$\langle e_0, (\tfrac{1}{2}MD)^{T-1} e_1\rangle \tag{45}$$

The generating function for the probabilities $P(A, T)$, defined as

$$Q(z, u)=\sum_{T, A} P(A, T)\, z^A (2u)^T \tag{46}$$

can therefore be expressed as

$$Q(z, u)=\left\langle e_0, \frac{2u}{1-uMD} e_1\right\rangle \tag{47}$$

since the Neumann series for the inverse of $(1-uMD)$ is easily seen to converge for $|u|\leqslant\frac{1}{2}$ and $|z|\leqslant 1$. The function Q is analogous to the function f in the previous section and we have placed a factor of 2 in front of the variable u in its definition for convenience.

Equation (47) allows us in principle to calculate $P(A, T)$. However, the interesting feature of $P(A, T)$ is that it falls exponentially with T and $P(A, T)=0$ unless $A\leqslant hT$. It follows that $P(A)=\sum_T P(A, T)$ falls exponentially with A provided A is large enough. In order to establish this exponential decay it suffices to show that $Q(1, u)$ is finite for some $u>\frac{1}{2}$. We can write

$$Q(1, u)=\frac{1}{2}\left\langle e_0, \frac{1}{\lambda-(1/2)\,M} e_1\right\rangle \tag{48}$$

where $\lambda=(2u)^{-1}$. Evaluating the matrix element in Eq. (48) we find

$$Q(1, u)=\frac{\cos((h-2)\,\theta)}{\cos((h-1)\,\theta)} \tag{49}$$

in the case $h\geqslant 3$, where

$$e^{i\theta}=\lambda+i\sqrt{1-\lambda^2} \tag{50}$$

and we are assuming $\lambda \leqslant 1$. The first singularity of $Q(1, u)$ as u moves beyond $\frac{1}{2}$ is encountered for the smallest $\theta \in [0, 2\pi)$ for which the denominator in Eq. (49) vanishes, i.e.,

$$\theta = \frac{\pi}{2(h-1)} \tag{51}$$

It follows that the radius of convergence of $Q(1, u)$ is

$$r = \frac{1}{2}\sqrt{1 + \tan^2 \frac{\pi}{2(h-1)}} \tag{52}$$

and for large A we find

$$P(A) \leqslant Ce^{-cA/h^2} \tag{53}$$

where c and C are positive constants and c can be taken to be independent of h.

The exponential decay of $P(A)$ takes over from the power law found in the previous section for $A \approx h^3$ since a random walk must have at least h^2 steps in order to feel the effect of the reflecting barrier at $y = h$. In order to prove this note that we can write

$$P(T) = \frac{2^{-T}}{2\pi i} \oint \frac{Q(1, u)}{u^{T+1}}\, du \tag{54}$$

where the contour encloses the unit disc in the complex plane. Calculating the residues we find that

$$P(T) = \frac{1}{2(h-1)} \sum_{n=0}^{2h-3} \sin^2 \frac{\pi(1+2n)}{2(h-1)} \cos^{T-2} \frac{\pi(1+2n)}{2(h-1)} \tag{55}$$

If Th^{-2} is very small then the sum over n can be approximated by a gaussian integral which gives $P(T) \sim T^{-3/2}$; thus we see that at large h $P(T)$ crosses over from exponential decay to $T^{-3/2}$ decay for $T \approx h^2$.

4. DISCUSSION

We do not expect our principal results to change if the simple Bernoulli random walk is replaced by a random walk with any rapidly decaying transition function. The exponent 4/3 and the exponential decay in the presence of a reflecting barrier ought to be universal. This is easily checked for a random walk with arbitrary α and β and also by numerical calculations in a few other simple cases.

We do, however, expect the power law to change for random walks with transition functions ϕ which do not decay rapidly (so that some higher moments of ϕ will diverge). In earthquakes the size distribution exponent for small and intermediate size events varies from around 2/3 to values greater than 1. Using the statistics of avalanches one can probably concoct a random walk model with an identical distribution but the more interesting problem of the correlations between different avalanches cannot be studied in this framework without allowing interacting random walks. The principal virtue of the model we have discussed is that it gives us a qualitative and quantitative insight into the genesis of the power law distribution for avalanches without introducing any complicated dynamics.

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